On the bandgap quantum coupler and the harmonic oscillator interacting with a reservoir: Defining the relative phase gate

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Abstract

In order to be able to study dissipation, the interaction between a single system and their environment was introduced in quantum mechanics. Master and quantum Langeving equations was derived and, also, decoherence was studied using this approach. One of the most used model in this field is a single harmonic oscillator interacting with a reservoir. In this work we solve analytically this problem in the resonance case with the evolution operator method. We use this result to study the conditional dynamics of a finite system of coupling, a bandgap quantum coupler. We study the conditional dynamics of the coupler on the computational basis by choosing a proper interaction time. This conditional dynamics provides a distinct realization of a quantum phase gate, which we name the relative phase gate.

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I. INTRODUCTION

One of the most important characteristics of the standard quantum theory is its restriction to be applied to closed systems, i.e. isolated of environmental influences, because the Scrödinger equation applies only to closed systems. If we take into account the environment we have an open system and the total Hamiltonian consists of the Hamiltonian of the open system, its environment (named reservoir or thermal bath too), and their interaction:

$$\hat{H}_{tot} = \hat{H}_{sys} + \hat{H}_{env} + \hat{H}_{int}. \tag{1}$$

Particularly, in quantum optics, it is of great practical importance the study of these systems. For instance, that the quantum system interacting with its environment becomes entangled with it is considered the profoundly quantum cause for decoherence. The most simple example of an open system is a simple harmonic oscillator interacting with an infinity set of harmonics oscillators. The corresponding hamiltonian, in the rotating-wave approximation,

$$\hat{H}_{tot} = w\hat{a}^{\dagger}\hat{a} + w\sum_{j=0}^{\infty}\hat{b}_j^{\dagger}\hat{b}_j + \sum_{j=0}^{\infty} g_j\left(\hat{a}\hat{b}_j^{\dagger} + \hat{a}^{\dagger}\hat{b}_j\right), \tag{2}$$

describing the resonant interaction between a single harmonic oscillator and the reservoir oscillators (we have taken $\hbar = 1$). The g_j is a real constant describing the linear coupling. In this case, the Schrödinger equation is regarded as unsolvable and the physical community (particularly the quantum opticians) have developed many valuable instruments to handle the problem. For instance, from equation (2) the following master equation in the Lindblad form at temperature $T \neq 0$ is obtained [1]:

$$\frac{\partial\hat{\rho}\left(t\right)}{\partial t} = i\omega_0\left[\hat{\rho}\left(t\right), \hat{a}^{\dagger}\hat{a}\right] + \frac{\gamma}{2}\left(\overline{n}+1\right)\left(2\hat{a}\hat{\rho}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger}\hat{a}\right) + \frac{\gamma}{2}\overline{n}\left(2\hat{a}^{\dagger}\hat{\rho}\hat{a} - \hat{a}\hat{a}^{\dagger}\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^{\dagger}\right), \tag{3}$$

where $\hat{\rho}\left(t\right)$ is the open system's reduced density operator. Equation (3) has been solved in many ways, see for example references [1, 2, 3, 4, 5, 6] and references therein. Another way to treat this problem is by using the Heisenberg-Langevin approach [5, 6], where the dynamical equations are deduced, also, from Hamiltonian (2). Both aproximations, i. e. the master and Langeving equations, focuses in the single system by tracing out the environmental degrees of freedom [1, 5, 6]. Additionally, the problem of dissipation is handled by introducing phenomenological decay constant [7, 8, 9].

On the other hand, the most straight way to solve the problem is by solving directly the time dependent Schrödinger's equation, i.e.

$$|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle\,, \tag{4}$$

where $\hat{U}(t) = e^{-\frac{it}{\hbar}\hat{H}_{tot}} = e^{-\frac{it}{\hbar}\left(\hat{H}_{sys}+\hat{H}_{env}+\hat{H}_{int}\right)}$ is called the time evolution operator [10, 11]. The common belief of equation (4) is that the determination of the solution is beyond to reach due to the difficult task to factorize $\hat{H}_{tot}$. In this work, we focus in the analytical solution of this problem by using the factorization method of the time evolution operator $\hat{U}(t)$ [10, 11] (also see reference [12]).

We apply this result to the study of conditional dynamics of a finite system of coupling. The system is an optical coupling device, called the bandgap quantum coupler, composed of a central waveguide surrounded by a finite number N of waveguides isolated one from another [13, 14]. In the limit case when $N \longrightarrow \infty$ the coupler approaches to an open system [13]. We propose a distinct realization of a quantum phase gate in the coupler, when $N = 2, 3$, by identified the optical modes of the waveguides with qubits and selecting a proper interaction time. We provide this quantum logic gate as an example of conditional quantum dynamics [15, 16].

II. AN HARMONIC OSCILLATOR INTERACTING WITH THE ENVIRONMENT

The coupling between a single harmonic oscillator and the environment is of great importance in quantum optics. It serves to model an open system and assist to study the causes of the loss of coherence. The most common way to determine the open system dynamics is by master equations

$$\frac{\partial\hat{\rho}(t)}{\partial t} = \hat{\pounds}\hat{\rho}(t)\,, \tag{5}$$

where $\hat{\pounds}$ denote the generator of this dissipative, non unitary dynamics. This equation is of so called Lindblad form, ensuring that general properties of density operator $\hat{\rho}(t)$ are preserved under time evolution. In this section we solve this problem using the evolution operator method [10, 11]. This method allow to find directly the evolution of a single harmonic oscillator coupled with the environment and avoids having to deal with the equation (5).

For a simple harmonic oscillator interacting with the environment the state system dynamics is given by:

$$|\Psi(t)\rangle = e^{\epsilon\left(\hat{H}_{sys}+\hat{H}_{env}\right)} e^{\epsilon \hat{H}_{int}} |\Psi(0)\rangle, \tag{6}$$

where $\epsilon \equiv -it$. Evidently the exponential terms are function of operators wich obey specific commutation rules. In order to obtain $|\Psi(t)\rangle$ we have to separate the exponential operators. For this purpose, we will use the factorization method to exponential functions of operators introduced in Refs. [10, 11]. From (2) immediately follows that $[\hat{H}_{sys}+\hat{H}_{env}, \hat{H}_{int}] = 0$ and only is necessary the factorization of interaction terms. Defining $\hat{L}_+ \equiv \epsilon \sum_{j=0}^{\infty} g_j \hat{a}^\dagger \hat{b}_j \equiv \hat{L}_-^\dagger$ and $\hat{L}_3 \equiv \frac{1}{2}\left(\sum_{j=0}^{\infty} \epsilon^2 g_j^2 \hat{a}^\dagger \hat{a} - \sum_{i,j=0}^{\infty} \epsilon^2 g_i g_j \hat{b}_i^\dagger \hat{b}_j\right)$ from the equation (2) we have

$$\left[\hat{L}_+, \hat{L}_-\right] = 2\hat{L}_3, \left[\hat{L}_3, \hat{L}_\pm\right] = \pm \sum_{j=0}^{\infty} \epsilon^2 g_j^2 \hat{L}_\pm. \tag{7}$$

The commutation relations (7) obey the standard angular-momentum commutation relations. In Refs. [10, 11] the problem has been solved making use of an unusual symmetrical structure on the exponential function of operators by repeating $\hat{L}_+$ or $\hat{L}_-$ on both sides of factorization array and by omitting the commutator $\hat{L}_3$. This result allow us do not treat with the commutator because the application of $\exp\left(\sum_{i,j=0}^{\infty} g_i g_j \hat{b}_i^\dagger \hat{b}_j\right)$ is a difficult task due to the exponential operator contains a double summatory. Therefore, using one of the two factorization forms found in Refs. [10, 11] for the unidimensional harmonic oscillator the equation (6) becomes:

$$|\Psi(t)\rangle = e^{\epsilon w\left(\hat{a}^\dagger \hat{a} + \sum_{j=0}^{\infty} \hat{b}_j^\dagger \hat{b}_j\right)} e^{\epsilon f(t) \sum_{j=0}^{\infty} g_j \hat{a}^\dagger \hat{b}_j} e^{\epsilon h(t) \sum_{j=0}^{\infty} g_j \hat{a} \hat{b}_j^\dagger} e^{\epsilon f(t) \sum_{j=0}^{\infty} g_j \hat{a}^\dagger \hat{b}_j} |\Psi(0)\rangle, \tag{8}$$

where

$$f(t) = \frac{1}{\sqrt{\gamma}} \tan\left(\sqrt{\gamma}/2\right), h(t) = \frac{1}{\sqrt{\gamma}} \sin\left(\sqrt{\gamma}\right), \tag{9}$$

and $\gamma \equiv \sum_{j=0}^{\infty} t^2 g_j^2$. There exist another factorization array for these operator algebras as it is shown in Refs. [10, 11] but we will use this array for simplicity in the application.

At this stage, we have solved formally the time dependent Schrödinger equation for the resonant case of a single harmonic oscilator interacting with a heat bath. Therefore to obtain the state system dynamics $|\Psi(t)\rangle$ is necessary to choose properly distinct initial states $|\Psi(0)\rangle$. In this work, we will not treat the problem of state system evolution under specific conditions on the initial states. Only we show the mathematical solution of this problem.

Instead, we will concentrate in the conditional quantum dynamics of a finite coupling system by choosing a specific interaction time and the initial state in the computation basis $\{0, 1\}$. This result will provide an example of a quantum logic gate for quantum computation. For an interesting discussion regarding the applicability of this model and a review of some methods to study it see the work of Presilla, Onofrio, and Tambini [9].

III. THE BANDGAP QUANTUM COUPLER

The bandgap quantum coupler is a device that emulate the interaction of a single harmonic oscillator with a finite set of oscillators. This device is an optical coupler formed from a central waveguide surrounded by a number finite N of waveguides isolated one from another, so that it is possible an interaction with only the central waveguide [13, 14]. We propose this device as a generator of a quantum logical gate by choosing a suitable interaction time. The Hamiltonian wich describes the system is given by

$$\hat{H} = w\hat{a}^{\dagger}\hat{a} + w\sum_{j=1}^{N}\hat{b}_j^{\dagger}\hat{b}_j + \sum_{j=1}^{N} g_j\left(\hat{b}_j\hat{a}^{\dagger} + \hat{a}\hat{b}_j^{\dagger}\right), \tag{10}$$

where $\hat{a}\left(\hat{a}^{\dagger}\right)$ and $\hat{b}_j\left(\hat{b}_j^{\dagger}\right)$ are the corresponding annihilation (creation) operators of the modes propagating in the central and jth waveguides. The $g_j = g\ (\forall\ j)$ is a real constant describing the linear coupling. Evidently, this hamiltonian is similar to that of equation (2), except for the limit of the summatory. Therefore we can obtain an immediate solution of the corresponding Schrödinger equation of the form:

$$|\Psi(t)\rangle = e^{\epsilon w\left(\hat{a}^{\dagger}\hat{a}+\sum_{j=1}^{N}\hat{b}_j^{\dagger}\hat{b}_j\right)} e^{\epsilon f(t)\sum_{j=1}^{N} g_j\hat{a}^{\dagger}\hat{b}_j} e^{\epsilon h(t)\sum_{j=1}^{N} g_j\hat{a}\hat{b}_j^{\dagger}} e^{\epsilon f(t)\sum_{j=1}^{N} g_j\hat{a}^{\dagger}\hat{b}_j} |\Psi(0)\rangle, \tag{11}$$

where $f(t)$ and $g(t)$ are the functions of the equation (9). In order to obtain a conditional dynamics wich implement a quantum logical gate in the coupler, we have to establish a connection between our system and the language of quantum computation. Bosonic qubits are defined by states of optical modes. An optical mode is a physical system whose state space consists of superpositions of the number states $|n\rangle$, where $n = 0, 1, ...$ gives the number of photons in the mode [17, 18]. Then, each optical mode in a waveguide represents a qubit and the corresponding $\hat{U}(t)$ generate the evolutions. Once we have made a suitable connection we will restrict our study to the case when $N = 1$ and the preparation of the initial state $|\Psi(0)\rangle$ in the two qubits computational basis $\{|0,0\rangle, |1,0\rangle, |0,1\rangle, |1,1\rangle\}$.

A. Relative phase gate on two qubits

Let us show how a two qubits phase gate can be performed using the interaction (11) when $N = 1$. This gate is a distinct realization of the known quantum phase gates [15]. Initially we consider the case $N = 1$, i.e. two waveguides, and prepare the initial state in the two qubits computational basis. We will use the notation $|n\rangle_a$ for the central waveguide and $|n\rangle_{b_1}$for first outer waveguide ($n = 0, 1$). For the particular interaction time $t = 2\pi/cg$ and $w = cg/2$ the sequence of final states of the interacting modes is given by

$$\begin{aligned} |0\rangle_a |0\rangle_{b_1} &\longrightarrow |0\rangle_a |0\rangle_{b_1} , \\ |1\rangle_a |1\rangle_{b_1} &\longrightarrow |1\rangle_a |1\rangle_{b_1} , \\ |1\rangle_a |0\rangle_{b_1} &\longrightarrow e^{i\pi} |1\rangle_a |0\rangle_{b_1} , \\ |0\rangle_a |1\rangle_{b_1} &\longrightarrow e^{i\pi} |0\rangle_a |1\rangle_{b_1} . \end{aligned} \tag{12}$$

The effect of this interaction is to change the phase if both qubits are in different states. Otherwise the qubits are left unchanged. From this result, we define the following conditional phase gate through the mathematical equation:

$$\hat{U}^{relative}_{Phase} |j_1\rangle |j_2\rangle = e^{i\pi(j_1 - j_2)} |j_1\rangle |j_2\rangle , \tag{13}$$

for $j_1, j_2 = 0, 1$. We will call $\hat{U}^{relative}_{Phase}$ as the *relative phase* gate [15, 19]. In summary we have made a new version of a quantum phase gate operating on the 2^2 quantum computational states. Below we give the conditional definition of this phase gate.

Before to analyze this new phase gate, let us ask: What kind of evolution is expected when a two-qubit phase gate acts on two unknown qubits?. That is, the question is: $\hat{U}^{two-qubit}_{Phase} (\alpha|0\rangle_1 + \beta|1\rangle_1) (\gamma|0\rangle_2 + \delta|1\rangle_2) \quad \rightarrow \quad ? \quad ?$. The definition of one-qubit phase gate can serve as a guide to answer this question, it is defined by its action on the one qubit computational basis states: if the qubit is in the state $|1\rangle$, then it applies a phase, that is $\hat{U}^{one-qubit}_{Phase}|1\rangle = e^{i\phi}|1\rangle$; and it does nothing when the qubit is in the state $|0\rangle$, that is $\hat{U}^{one-qubit}_{Phase}|0\rangle = |0\rangle$ [15]. Therefore, when the one-qubit phase gate acts on an unknown qubit, it produces the evolution: $\hat{U}^{one-qubit}_{Phase} (\alpha|0\rangle + \beta|1\rangle) = (\alpha|0\rangle - \beta|1\rangle)$. Then, from this definition, we can infer that a two-qubit phase gate must induce the following conditional evolution:

$$\hat{U}^{two-qubit}_{Phase} (\alpha|0\rangle_1 + \beta|1\rangle_1) (\gamma|0\rangle_2 + \delta|1\rangle_2) = (\alpha|0\rangle_1 - \beta|1\rangle_1) (\gamma|0\rangle_2 - \delta|1\rangle_2) \tag{14}$$

Now, let us review some properties of others two-qubit phase gates defined in the literature [15]. As far as we know, there are two two-qubits phase gates defined. The first case, which is called the control C phase gate, or Cirac-Zoller phase gate [20], $\hat{U}^{C}_{Phase}$, is defined as: *If the control qubit is set to* $|1\rangle$*, then apply the one-qubit phase gate to the target qubit. Otherwise, if the control qubit is set to* $|0\rangle$*, then left the target qubit unchanged.* This definition produces the following evolution: $\hat{U}^{C}_{Phase}|0\rangle_1|0\rangle_2 = |0\rangle_1|0\rangle_2$, $\hat{U}^{C}_{Phase}|0\rangle_1|1\rangle_2 = |0\rangle_1|1\rangle_2$, $\hat{U}^{C}_{Phase}|1\rangle_1|0\rangle_2 = |1\rangle_1\hat{U}^{one-qubit}_{Phase}|0\rangle_2 = |1\rangle_1|0\rangle_2$, $\hat{U}^{C}_{Phase}|1\rangle_1|1\rangle_2 = |1\rangle_1\hat{U}^{one-qubit}_{Phase}|1\rangle_2 = |1\rangle_1\left(-|1\rangle_2\right) = -|1\rangle_1|1\rangle_2$. In short:

$$\hat{U}^{C}_{Phase}|m\rangle_1|n\rangle_2 = |m\rangle_1\left(e^{imn\pi}|n\rangle_2\right), \tag{15}$$

where, $m, n = 0, 1$.

In the second case, which we call the control phase shift gate $\hat{U}^{shift}_{Phase}$, the definition given on page 180 of reference [15] is as follows: *If the control qubit is set to* $|0\rangle$*, then the target qubit is left alone. Otherwise, if the control qubit is set to* $|1\rangle$*, then apply a phase shift to the target qubit.* This definition produces the following evolution $\hat{U}^{shift}_{Phase}|0\rangle_1|0\rangle_2 = |0\rangle_1|0\rangle_2$, $\hat{U}^{shift}_{Phase}|0\rangle_1|1\rangle_2 = |0\rangle_1|1\rangle_2$, $\hat{U}^{shift}_{Phase}|1\rangle_1|0\rangle_2 = |1\rangle_1\left(-|0\rangle_2\right) = -|1\rangle_1|0\rangle_2$, $\hat{U}^{shift}_{Phase}|1\rangle_1|1\rangle_2 = |1\rangle_1\left(-|1\rangle_2\right) = -|1\rangle_1|1\rangle_2$. In short:

$$\hat{U}^{shift}_{Phase}|m\rangle_1|n\rangle_2 = |m\rangle_1\left(e^{im\pi}|n\rangle_2\right), \tag{16}$$

where, $m, n = 0, 1$. Note that the phase induced by this gate depends exclusively on the value m of the control qubit. Other characteristic of this phase gate is that, contrary to the one-qubit phase gate $U^{one-qubit}_{Phase}$, it applies a phase to the base state $|0\rangle$.

On the other hand, the control phase shift gate $\hat{U}^{shift}_{Phase}$ suggest, seemingly, a change on the target qubit. However, when this phase gate is applied to two unknown qubits it produces the following change:

$$\hat{U}^{shift}_{Phase}\left(\alpha|0\rangle_1 + \beta|1\rangle_1\right)\left(\gamma|0\rangle_2 + \delta|1\rangle_2\right) = \left(\alpha|0\rangle_1 - \beta|1\rangle_1\right)\left(\gamma|0\rangle_2 + \delta|1\rangle_2\right). \tag{17}$$

Equation (17) implies $\hat{U}^{shift}_{Phase}|0\rangle_1\left(\gamma|0\rangle_2 + \delta|1\rangle_2\right) = |0\rangle_1\left(\gamma|0\rangle_2 + \delta|1\rangle_2\right)$, and $\hat{U}^{shift}_{Phase}|1\rangle_1\left(\gamma|0\rangle_2 + \delta|1\rangle_2\right) = -|1\rangle_1\left(\gamma|0\rangle_2 + \delta|1\rangle_2\right)$. It seems as if this gate produces a phase change only in the control qubit.

Also, when we apply the control C phase gate on two unknown qubits we obtain the following entangled state:

$$\hat{U}^{C}_{Phase}\left(\alpha|0\rangle_1 + \beta|1\rangle_1\right)\left(\gamma|0\rangle_2 + \delta|1\rangle_2\right) = \alpha|0\rangle_1\left(\gamma|0\rangle_2 + \delta|1\rangle_2\right) + \beta|1\rangle_1\left(\gamma|0\rangle_2 - \delta|1\rangle_2\right). \tag{18}$$

Equation (18) implies that $\hat{U}^{C}_{Phase}|0\rangle_1 \left(\gamma|0\rangle_2 + \delta|1\rangle_2\right) = |0\rangle_1 \left(\gamma|0\rangle_2 + \delta|1\rangle_2\right)$, and that $\hat{U}^{C}_{Phase}|1\rangle_1 \left(\gamma|0\rangle_2 + \delta|1\rangle_2\right) = |1\rangle_1 \left(\gamma|0\rangle_2 - \delta|1\rangle_2\right)$. That is, the control C phase gate induces a phase change only when the first qubit is in the state $|1\rangle_1$ and the second is in an unknown state. Because its ability to produce entangled sates, the control C phase gate (and its three qubit generalization) is the most studied [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32] of the phase gate's family.

On the other hand, the relative phase gate, $\hat{U}^{relative}_{Phase}$, can be stated in conditional sentence of "If-Then" form as follows:

> DEFINITION 1. If the states of a two parties system are not equal, then apply the one-qubit phase gate, $U^{one-qubit}_{Phase}$, on each qubit. Otherwise, left them unchanged.

Where, by equal states we means: $|0\rangle_1|0\rangle_2$ or $|1\rangle_1|1\rangle_2$. It is worthwhile to note that the conditional property of this phase gate is given by the whole state of the system, which represents a conditional evolution that depend on the overall state of the whole system and induces a conditional evolution on each subsystem, see reference [33]. DEFINITION 1 produces the following conditional evolution on the computational basis of two qubits $\Big($here we let the one-qubit phase gate to apply an arbitrary phase θ, that is $\hat{U}^{one-qubit}_{Phase}|1\rangle = e^{i\theta}|1\rangle\Big)$: $\hat{U}^{relative}_{Phase}|0\rangle_1|0\rangle_2 = |0\rangle_1|0\rangle_2$, $\hat{U}^{relative}_{Phase}|1\rangle_1|1\rangle_2 = |1\rangle_1|1\rangle_2$, $\hat{U}^{relative}_{Phase}|0\rangle_1|1\rangle_2 = \hat{U}^{one-qubit}_{Phase}|0\rangle_1\hat{U}^{one-qubit}_{Phase}|1\rangle_2 = e^{i\theta}|0\rangle_1|1\rangle_2$, $\hat{U}^{relative}_{Phase}|1\rangle_1|0\rangle_2 = \hat{U}^{one-qubit}_{Phase}|1\rangle_1\hat{U}^{one-qubit}_{Phase}|0\rangle_2 = e^{i\theta}|1\rangle_1|0\rangle_2$.

Now, when we apply the relative phase gate $\hat{U}^{relative}_{Phase}$ to two unknown qubits we obtain the following entangled state:

$$\hat{U}^{relative}_{Phase}\left(\alpha|0\rangle_1 + \beta|1\rangle_1\right)\left(\gamma|0\rangle_2 + \delta|1\rangle_2\right) = \alpha|0\rangle_1\left(\gamma|0\rangle_2 + \delta e^{i\theta}|1\rangle_2\right) + \beta e^{i\theta}|1\rangle_1\left(\gamma|0\rangle_2 + \delta e^{-i\theta}|1\rangle_2\right) \tag{19}$$

For the particular case $\theta = \pi$ we obtains:

$$\hat{U}^{relative}_{Phase}\left(\alpha|0\rangle_1 + \beta|1\rangle_1\right)\left(\gamma|0\rangle_2 + \delta|1\rangle_2\right) = \left(\alpha|0\rangle_1 - \beta|1\rangle_1\right)\left(\gamma|0\rangle_2 - \delta|1\rangle_2\right). \tag{20}$$

Furthermore, equation (20) implies that $\hat{U}^{relative}_{Phase}|0\rangle_1\left(\gamma|0\rangle_2 + \delta|1\rangle_2\right) = |0\rangle_1\left(\gamma|0\rangle_2 - \delta|1\rangle_2\right)$ and $\hat{U}^{relative}_{Phase}|1\rangle_1\left(\gamma|0\rangle_2 + \delta|1\rangle_2\right) = -|1\rangle_1\left(\gamma|0\rangle_2 - \delta|1\rangle_2\right)$. Form this result, we can say that the $\hat{U}^{relative}_{Phase}$ gate induce a phase change when acts on two unknown qubits.

It is interesting to note that the relative phase gate in equation (20) can be implemented using the control phase shift and SWAP gates, i. e. $\hat{U}^{relative}_{Phase} = \hat{U}^{shift}_{Phase}\hat{U}_{SWAP}\hat{U}^{shift}_{Phase}\hat{U}_{SWAP}$.

B. Three relative phase gate

In the previous section we have proposed a distinct phase gate $\hat{U}^{relative}_{Phase}$ in terms of a conditional dynamics on the global state system. Moreover, a serie of results concerning the classification of two qubits phase gates from conditional dynamics were presented. In the following we will generalize the result of the $\hat{U}^{relative}_{Phase}$ gate to the case of a three qubits gate. Consider now the case $N = 2$, i.e. three waveguides, in equation (10) and the initial state $|\Psi(0)\rangle$ prepared in the three qubits computational basis $\{|0,0,0\rangle, |1,0,0\rangle, |0,0,1\rangle, |0,1,0\rangle, |1,1,0\rangle, |1,0,1\rangle, |0,1,1\rangle, |1,1,1\rangle\}$. We will use the notation $|n\rangle_a$ for the central waveguide and $|n\rangle_{b_1} |n\rangle_{b_2}$ for the first and second outer waveguides $(n = 0, 1)$. For the specific interaction time $t = 2\pi/cg$ and $w = cg/2$ the sequence of final states of the interacting modes is given as follows:

$$\begin{aligned}
|0\rangle_a |0\rangle_{b_1} |0\rangle_{b_2} &\longrightarrow |0\rangle_a |0\rangle_{b_1} |0\rangle_{b_2}, \\
|0\rangle_a |1\rangle_{b_1} |1\rangle_{b_2} &\longrightarrow |0\rangle_a |1\rangle_{b_1} |1\rangle_{b_2}, \\
|1\rangle_a |0\rangle_{b_1} |1\rangle_{b_2} &\longrightarrow |1\rangle_a |0\rangle_{b_1} |1\rangle_{b_2}, \\
|1\rangle_a |1\rangle_{b_1} |0\rangle_{b_2} &\longrightarrow |1\rangle_a |1\rangle_{b_1} |0\rangle_{b_2}, \\
|1\rangle_a |0\rangle_{b_1} |0\rangle_{b_2} &\longrightarrow e^{i\pi} |1\rangle_a |0\rangle_{b_1} |0\rangle_{b_2}, \\
|0\rangle_a |1\rangle_{b_1} |0\rangle_{b_2} &\longrightarrow e^{i\pi} |0\rangle_a |1\rangle_{b_1} |0\rangle_{b_2}, \\
|0\rangle_a |0\rangle_{b_1} |1\rangle_{b_2} &\longrightarrow e^{i\pi} |0\rangle_a |0\rangle_{b_1} |1\rangle_{b_2}, \\
|1\rangle_a |1\rangle_{b_1} |1\rangle_{b_2} &\longrightarrow e^{i\pi} |1\rangle_a |1\rangle_{b_1} |1\rangle_{b_2}.
\end{aligned} \tag{21}$$

The effect of the interaction is to change the sign of the global state if one or all qubits are in the first excitated state $|1\rangle$. Otherwise the qubits are left unchanged. From this result, we represent the conditional three qubits phase gate through the mathematical equation:

$$\hat{U}^{relative}_{Phase} |j_1\rangle |j_2\rangle |j_3\rangle = e^{i\pi(j_1 - j_2 - j_3)} |j_1\rangle |j_2\rangle |j_3\rangle, \tag{22}$$

for all $j_1, j_2, j_3 = 0, 1$. The three qubits relative phase gate, $\hat{U}^{relative}_{Phase}$, can be stated in the form of a conditional sentence "If-Then" as follows:

> DEFINITION 2. If the states of a three parties system are not in the basic sate, i. e. $|0\rangle_1 |0\rangle_2 |0\rangle_3$ or an equivalent state, then apply the one-qubit phase gate, $U^{one-qubit}_{Phase}$, on each qubit. Otherwise, left them unchange

In this sense, we have made a conditional property of the three qubits $\hat{U}^{relative}_{Phase}$ on the whole state of the system, i.e. a condition on the global sate of the system through the values of j_1, j_2 and j_3.

Now, when we apply the three qubits relative phase gate $\hat{U}^{relative}_{Phase}$ to three unknown qubits we obtain the following result:

$$\begin{aligned} &\hat{U}^{relative}_{Phase} \left(\alpha|0\rangle_1 + \beta|1\rangle_1\right)\left(\gamma|0\rangle_2 + \delta|1\rangle_2\right)\left(\eta|0\rangle_3 + \zeta|1\rangle_3\right) \\ &\quad = \left(\alpha|0\rangle_1 - \beta|1\rangle_1\right)\left(\gamma|0\rangle_2 - \delta|1\rangle_2\right)\left(\eta|0\rangle_3 - \zeta|1\rangle_3\right), \end{aligned} \tag{23}$$

according to equation (22). The equation (23) implies that $|0\rangle_1 \left(\gamma|0\rangle_2 + \delta|1\rangle_2\right)\left(\eta|0\rangle_3 + \zeta|1\rangle_3\right) \rightarrow |0\rangle_1 \left(\gamma|0\rangle_2 - \delta|1\rangle_2\right)\left(\eta|0\rangle_3 - \zeta|1\rangle_3\right)$ and $|1\rangle_1 \left(\gamma|0\rangle_2 + \delta|1\rangle_2\right)\left(\eta|0\rangle_3 + \zeta|1\rangle_3\right) \rightarrow e^{i\pi}|1\rangle_1 \left(\gamma|0\rangle_2 - \delta|1\rangle_2\right)\left(\eta|0\rangle_3 - \zeta|1\rangle_3\right)$. From this result, we can say that the three qubits $\hat{U}^{relative}_{Phase}$ gate induce a phase change when acts on three unknown qubits, as it was expected from the extension of two qubits $\hat{U}^{relative}_{Phase}$ gate.

IV. CONCLUSIONS

In this work we have addressed the problem of interaction between an harmonic oscillator and a reservoir. We have found an analytical solution to the Schrödinger equation in the resonant case. Also, we have presented an explicit example of a finite coupling device wich simulate the dynamics of an open system in the limit $N \rightarrow \infty$. The coupler is an optical device of waveguides concentrated around a central waveguide. Such a configuration allow us to find particular applications by choosing properly the interaction time. In the cases $N = 2, 3$ we have found a conditional dynamics wich provides an example of a quantum logic gate. This gate adhere a phase on the global state according to a conditional logic operation. This gate let us to find important differences with the control C phase gate and the control shift phase gate. We believe that this result may be useful to establish a classification of two qubits logic gates in two classes. Gates wich act on a single qubit and gates wich act on two qubits depending on a conditional operation [33].

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